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Evaluation of growth functions for tree height modelling

Vergleich von Höhenzuwachsfunktionen

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Abstract

Several growth functions (e.g. Backman, Bertalanffy, Evolon, Fischer, Gomperz, Gram, Grosenbaugh, Hassell, Hill, Hoßfeld, Hyperlogistic, Johnson, Koller, Korf, Kosun, Kövessi, Kumaraswamy, Levakovic, Logit, Maynard, Michaelis-Menten, Michailoff, Mitscherlich, Morgan, Nelder, Peschel, Richards, Ricker, Robertson, Schnute, Schumacher, Siven, Sloboda, Strand, Stannard, Terazaki, Thomasius, Todorovic, Yoshida, Weber, Weilbull) have been compared for their ability to follow observed height developments. Those with two coefficients might not be flexible enough to follow possible growth patterns. Some functions with three and especially those with four parameters can follow a wide range of growth patterns. The differences between most of the four

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parameter functions are marginal. Choosing one can depend on their behaviour to converge during parameterisation or the direct interpretability of the parameters selection. Overall the function h = c0 * (ln(1 + c2 * t**c3))**c1 showed good behaviour and might be one which could be recommended.

Zusammenfassung

Es wurden einige Wachstumsfunktionen (z.B. Backman, Bertalanffy, Evolon, Fischer, Gomperz, Gram, Grosenbaugh, Hassell, Hill, Hoßfeld, Hyperlogistic, Johnson, Koller, Korf, Kosun, Kövessi, Kumaraswamy, Levakovic, Logit, Maynard, Michaelis-Menten, Michailoff, Mitscherlich, Morgan, Nelder, Peschel, Richards, Ricker, Robertson, Schnute, Schumacher, Siven, Sloboda, Strand, Stannard, Terazaki, Thomasius, Todorovic, Yoshida,Weber,Weilbull) hinsichtlich ihrer Fähigkeit, beobachtete Höhenwachstumsverläufe zu beschreiben, verglichen. Jene mit zwei Koeffizienten sind zu unflexibel um das Spektrum an möglichen Wachstumsverläufen abzudecken. Funktionen mit drei und insbesondere jene mit vier Koeffizienten können ein breites Spektrum an-Wachtumsverläufen abdecken. Die Unterschiede zwischen den meisten Vierparameterfunktionen sind minimal, wodurch deren Auswahl auf andere Kriterien, wie Konvergierfreudigkeit oder direkte Interpretierbarkeit der Parameter, verlagert werden kann. Generell zeigte die Funktion h = c0 * (ln(1 + c2 * t**c3))**c1 gute Resultate und kann zur Beschreibung von Höhenverläufen empfohlen werden.

1. Introduction

Site index curves are used to estimate the productivity of a forest at a specific site. In the past those curves have been drawn freehand. Later, averages for specific site and age classes have been calculated and the resulting points have been connected. Nowadays those curves are typically drawn by using a function. This function should be flexible enough to reproduce the variety of observed height growth patterns. In this article established functions are compared as to how well they fit to observed height developments over time.

In the past many growth functions have been published and also several comparisons of some of them have been done (Levakovic, 1935; Peschel, 1938; Todorovic, 1961; Zeide, 1993; Ricker, 1979; Elfving and Kiviste, 1997; Zhang, 1997; Chrobok et al, 2004; Palah et al, 2004; Khamis et al, 2005; Upadhyay et al, 2005; Koya and Goshu, 2013; Kuelka and Maruk, 2015; Sedmk and Scheer, 2015). It appears that there was a dissatisfaction with the older growth functions or uncertainties which one to select. For the same reasons, the motivation comes to write this article. It will give a collection of more than 50 growth functions and show how they are related to each other. It will show the typical root–mean–square error (RMSE), the bias at some age steps and if there is a trend over site index between observed tree heights and each single growth function and also how the parameterized growth function will predict heights over an age of up to 800 years.

2. Materials and methods

2.1 Observed tree height development

Data-sets from von Guttenberg (1915) and stem analysis from BFW (Austrian Research and Training Centre for Forests, Natural Hazards and Landscape) have been used. The analysed trees (Picea abies) were raised in Austria. The used trees must have a minimum age of 100 years and should not have been suppressed by higher trees. The height–age curves of these trees are used as they are and will not be identical with corresponding site index curves which are typical created by smoothing many observations of dominant or mean heights for many ages. In total, 81 trees from Guttenberg and 95 trees from BFW have been taken. Figure 1 shows the height development of these trees. In the following compilation the distribution of age and height at the age of 50 years (h50) and 100 years (h100) is shown.

	Src	Min.	25 %	50 %	Mean	75 %	Max.
Age	Gut	100.0	120.0	140.0	133.2	150.0	150.0
	BFW	100.0	108.0	129.0	129.9	143.0	202.0
h50	Gut	2.70	9.30	12.90	13.27	17.50	23.20
	BFW	5.45	11.12	14.84	14.94	19.00	27.20
h100	Gut	7.70	17.90	23.40	23.51	29.10	37.00
	BFW	10.97	22.06	24.74	25.89	31.08	36.71

2.2 Growth functions

In literature many growth functions can be found. Also different requirements, which the function should fulfil, are stated. Returning a size of zero at time zero seems to be a good behaviour of the function. But in most cases it might be good enough if it returns something close to zero at time zero. Also the question arises when is time zero. Is it zero when the egg cell is built, when it is fertilised, when the seed is distributed, when the seed starts to germinate, … ? There is no need that it returns a finite number at infinite time, no need to be asymptotic, as this time is never reached. It is good enough when the returned sizes at ages, which could be maximally reached, are still in a reasonable range. The conditions that the increments and also the slope of the increments at time zero have to be zero are questionable and could be determined by the observations. Also a nice behaviour is if the function returns an equal or larger value for an increased time. If it shows a peak, the size after this peak could be kept at this peak for older ages. Here the functions are used as they are. So no modifications afterwards occur.

The growth function should have an inflection point because height increment typically shows a maximum. Growth functions can be distinguished in functions which estimate the actual size at a certain point in time and functions which estimate the change of size at a given time or size. Those types can be transformed to each other by integration and differentiation respectively. The change of size can be estimated absolutely or relatively (e.g. change in percentages of current size). In the past many authors have done research in this field, so only a selection of popular functions could be made. One of the simplest equation will be the linear function (eq. 1.0) where *h* is the tree height, *t* is the age and c_0 is a coefficient. Here the size is linear increasing with time. It will represent the lower benchmark which should be exceeded in most cases by all other functions. A pragmatic extension can be the additional use of t^2 like in function 1.1 which will show a maximum. This function could be extended further with some polynomials as shown in $h = c_0(1 - e^{c_1 \cdot t^{-1}})^{c_2}$ function 1.2 and 1.3. Another way to increase the curve flexibility can be reached by powering the age with a coefficient as shown in function 1.4 which is equivalent to . Polynomial and allometric can also be combined as shown in function 1.5.

Another way would be to use concave monotonically increasing functions and place exponents on possible places like shown for ln in function 2.2, for atan in function 2.0 and for tanh in function 2.1.

Terazaki (1915) introduced function 3.0 which was reinvented e.g. by Johnson (1935), Schumacher (1939) and Michailoff (1943). This function can be extended by adding the term $c_2 \cdot t^{-0,5}$ as shown in function 3.1 or by estimating the exponent with a coefficient as shown in function 3.2. The function of Korf (1939) is an extended version of Terazakis function as the fixed exponent –1 is exchanged by a coefficient (function 3.3). Function 3.4 from Gompertz (1825) looks again similar to function 3.3. Only the term t^{c_2} is exchanged with $e^{c_2 \times t}$. $h = \frac{c_0}{(1+c_1 \cdot e^{c_2 \cdot t})^{c_3}}$ was published by Stannard et al (1985) which produces quasi equivalent results like eq. 3.4. Sloboda (1971) again extended function 3.4 by introducing an exponent to t which results in function 3.5. Grosenbaugh (1965) published the function $h = c_0 + c_1 \cdot (e^{(c_2^2-1) \cdot U} - c_2 \cdot U)^{c_2 \cdot c_3 + 1}$ where U is a function such as $U = e^{c_4 \cdot t}$. His function is able to describe growth like a couple of other growth functions. This is possible by selecting specific values for some coefficients and using different functions for the parameter U. For instance by setting $c_2 = 0$, $c_0 = 0$ and using $U = c_4 \cdot e^{c_5 \cdot t^{c_6}}$ it will be exactly the same like function 3.5. In the tested function 3.6 the first term was eliminated, $U = e^{c_4 \cdot t^{c_5}}$ was used and the coefficient numbers have been reassigned. Function 3.7 was used by Korsun

(1935) which is similar to function 8.0 from Backman (1931) with the difference that one estimates the height, the other the height increment. This function has been extended, as shown in function 3.8 and 3.9 by adding another element to the polynomial or by exchanging the fixed exponent by a coefficient.

Weber (1891) has propagated function 4.0. Weber (1891) skipped the first years and applied then his function. Here this function is used for all ages. Pütter (1920) and von Bertalanffy (1934) have used $h = c_0(1 - c_2 \cdot e^{c_1 \cdot t})$, which also was called "monomolecular", according to the speed of this chemical reaction. Tests of this function ended with the result that most of the time $c_2 = 1$ was estimated and in cases it was not 1 the function returns a height unequal zero at age zero. This function is sometimes also named after Mitscherlich (1909). There are several ways to extend function 4.0. One is to combine some of these functions by multiplication as shown in function 4.1 and 4.2. It is also possible to add an exponent as shown in function 4.3. Kövessi (1928) joined two functions of Weber together by adding them as shown in equation 4.4. Kövessi (1929) reduced his function to a three coefficient form as shown in equation 4.5. By adding an exponent to Webers function, function 4.6 will be the result which was introduced by Mitscherlich (1919). If this exponent is set to $c_2 = 3$ the function was sometimes named after "Bertalanffy". This exponent $c_2 = 3$ comes from transforming observed length into weight. Schnute (1981) showed a function $h = (y_1^b + (y_2^b - y_1^b) \cdot \frac{1-e^{a*(t-t_1)}}{1-e^{a*(t_2-t_1)}})^{(1/b)}$ which can be transferred to Mitscherlich's function if the two selectable ages are set to $t_1 = 0$ and $t_2 = \infty$ and the sizes (y) at these ages are $y_1 = 0$ and y_2 gets the highest possible height. Richards (1959) added to this function an additional factor $h = c_0(1 - c_4 \cdot e^{c_1 \cdot t})^{c_2}$ where c4 sets the size at time zero depending on the the coefficients. Tests of this function ended with the result that most of the time $c_4 = 1$ was estimated and when it was not 1 the function returns a height unequal zero at age zero. Chapman (1931) also used this function which was then named the Chapman–Richards–Function. Function 4.0 could also be extended by adding an exponent to t which is shown in function 4.7 (Fisher and Tippett, 1928). In $h = c_0(1 - c4 \cdot e^{c_1 \cdot t^{c3}})$ the coefficient c_4 is added to eq. 4.7 which is called after Weilbull. Tests of this function ended with the result that $c_4 = 1$ was most of the time estimated and in cases it was not 1 the function returns a height unequal zero at age zero. $h = c_0(1 - (1 - (\frac{t}{c_3})^{c_1})^{c_2})$ was developed by Kumaraswamy (1980) which converges to eq. 4.7 if c_3 has high values. Coefficient estimates for the used data set showed high values for c_3 and estimates close to identical to those from eq. 4.7. Todorovic (1961) extended this function type to four coefficients which leads to function 4.8. Another extension of Weber's function was done by Thomasius (1964) and is given in function 4.9.

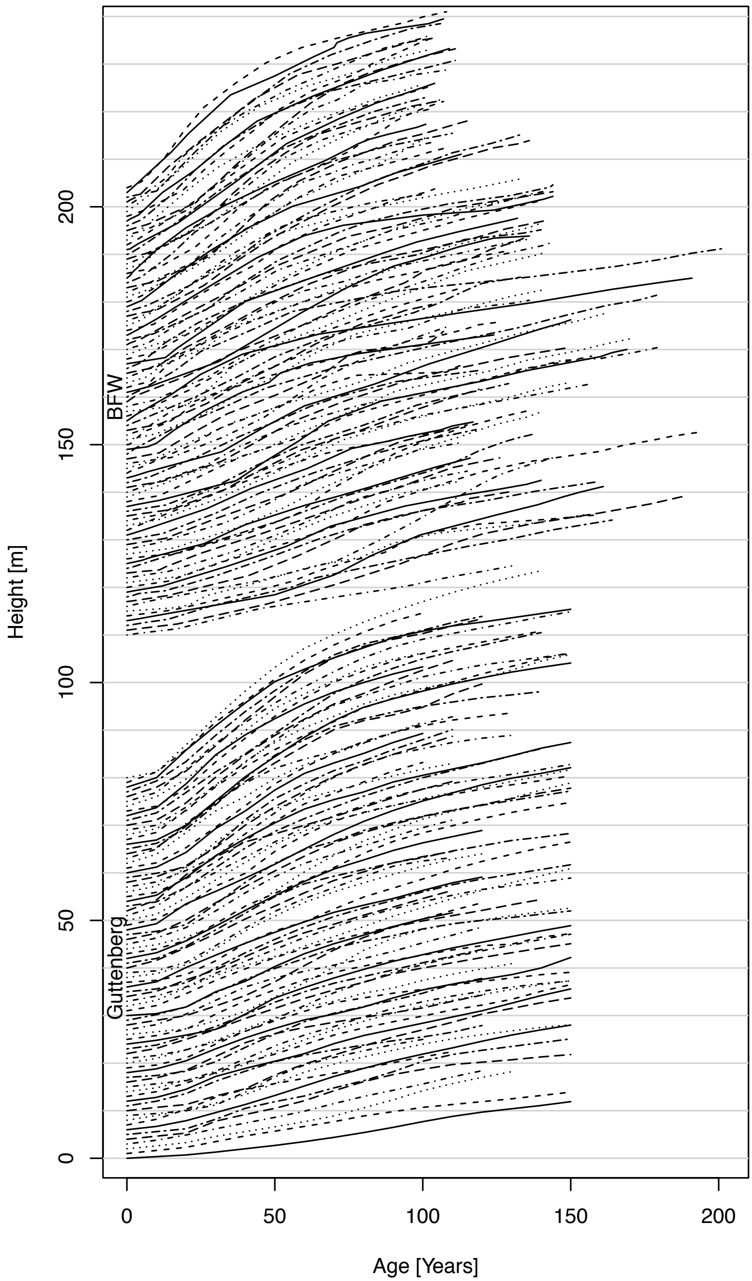

Figure 1: Height development of the trees: The height at age 0 is shifted from tree to tree by one meter.

The functions 5.1, 5.0 and 5.4 are from Hoßfeld (1822). Equation 5.2 shows a simple extension of function 5.1 by adding the term $c_3 \cdot t^{-3}$. The function from Morgan et al (1975) is equivalent to 5.4 and could be alternatively formulated as $h = \frac{c_0}{1+c_1 \cdot e^{c_2 \cdot \ln(t)}}$. Also the functions by Hill (1913), Michaelis and Menten (1913), Smith and Slatkin (1973) or Hassell (1975) are equal to 5.4. Yoshida (1928) exchanged the term $c_2 \cdot t^{-2}$ from Hoßfeld1 (eq. 5.1) with $c_2 \cdot t^{c_3}$ as shown in function 5.3. Strand (1964) has set the exponent $c_2 = -1$ of Hoßfeld4 (eq. 5.4) and powered the denominator by 3 (eq. 5.5). Levakovic (1935) already generalised the function of Strand by exchanging the exponent 3 with a coefficient (eq. 5.6) and in a further step he exchanged the exponent –1 also with a coefficient (eq. 5.7). This function was reinvented e.g. by Gottschalk and Dunn (2005). In this function group (eq. 5.0 – 5.7) typically a division by zero at time zero is produced, so it would sometimes help to reform those functions. $h = c_0 \cdot (1 - \frac{1}{1+c_1 \cdot t^{c_2}})^{c_3}$ will produce the same like Levakovic159 (eq. 5.7) but avoiding the division by zero.

The logit function is shown in eq. 6.0 which is identical with the function by Robertson (1908) and named "autocatalytic" as the speed of this chemical reaction can be described with it. The basic form of eq. 6.0 could be extended by adding an exponent to age as shown in eq. 6.1. Alternatively also a polynomial could be used as shown in eq. 6.2 with the untransformed age or in eq. 6.3 with the logarithm of age. Nelder (1962) added an exponent to the whole denominator as shown in eq. 6.4.

Sivèn (1896) has estimated tree height with $h = h_x \cdot (\frac{t}{t_x})^{0.25 \cdot \sqrt{t/t_x}}$ where h_x is the height at time t_x. A generalised form is given in function 7.0. Gram (1879) uses function 7.1 to estimate height over age which is similar to Ricker (1954). Peschel (1938) has formulated function 7.2 based on some predefined assumptions.

The previous functions describe the absolute size at a given time. The following functions describe the change of size at a given time. Backman (1931) has used function 8.0 to describe also tree height growth. For this purpose he has used this function in a series of up to three functions and combined their results either by adding or by multiplying the sub-function results. Here this combination was not made. This function was extended by adding the term $\ln^4(t)$ as shown in equation 8.1. Koller (1886) has used function 8.2 which was extended by Levakovic (1935) as shown in equation 8.3.

The hyperlogistic differential equation was formulated as $ih = c_0 \cdot (c_1 + h)^{c_2} \cdot (c_3 - (c_1 + h)^{c_4})^{c_5}$ which is based on Euler's beta function and shows the change of size for a given size. It is based on works by Pearson (1934), Richards (1959), Nelder (1961), Blumberg (1968), Turner et al (1969), Turner et al (1976) or Buis (1991). It can be assumed, that only two of the three exponents are significant and so typically $c_4 = 1$. This function has been used by Mende and Albrecht (2001) to describe a height curve and named the function Evolon-Model. The form shown in function 9.0 was used for coefficient estimation. In function 9.0 the coefficient c_1 is needed to allow an increment estimation different from zero when $h = 0$. The same problem can be solved with function 9.1. If an upper limit is not needed or wanted, alternatively $e^{c_3 . h^{c_4}}$ can be used instead of $(1 - h/c_3)^{c_4}$ which is shown in function 9.2 which is similar to the formulation of Gram (eq. 7.1), Koller (eq. 8.2) or Levakovic147 (eq. 8.3). Also Zeide (1993) has published similar functions. Function 9.3, 9.4 and 9.5 show additional examples how the second term could be exchanged with $1/(1+c_3 . h^{c_4})$, $1 - \tanh(c_3 . h^{c_4})$ and $\Pi/2 - \mathrm{atan}(c_3 . h^{c_4})$, respectively. To those terms an additional exponent could be added like $1/(1 + c_3 . h^{c_4})^{c_5}$, $1 - \tanh^{c_5}(c_3 . h^{c_4})$ and $\Pi/2 - \mathrm{atan}^{c_5}(c_3 . h^{c_4})$. It could also be tried to exchange the term $c_0 + c_1 . h^{c_2}$ by some other function or to combine the accelerator and decelerator by a subtraction instead of a multiplication.

It also might be possible to use some of the functions previously shown to describe the increment at a specific size simply by exchanging time with size. In the equation from Backman (eq. 8.0) and Korsun (eq. 3.7) or Gram (eq. 7.1) and Koller (eq. 8.2) something similar is done, as the same function describes once height and some other time height increment.

During the comparison it turned out that eq. 2.2 showed good results. So this function was further examined. To facilitate the parameter estimation c2 can be exchanged with e^{c_2} and the function is formulated as $h = c_0 . ln^{c_1}(1 + e^{c_2} . t^{c_3})$. In the next step it was tried to keep coefficients constant for each of the the two data-set. In eq. 2.3 c_1 and in eq. 2.4 c_3 was kept constant. It was also tried to keep both coefficients c_1 and c_3 constant (eq. 2.5). It was also tested if this function, with two varying parameters, can be improved by allowing a linear dependency of c_3 on c_2 (eq. 2.6).

2.3 Height curve parameter estimation

The function coefficients are estimated with nonlinear least square methods for each single tree using R (Core Team, 2015). Such methods need predefined starting parameters from which their coefficient optimisation begins. Bad starting parameters can lead to sub-optimal results. Finding one set of starting parameters which will work for each single tree might not be possible in each case. When it is possible to linearize the equation, individual starting coefficients for each tree are used. It is also possible to try a range of different starting parameters and select those which have shown the best result. The starting parameter search was done with the function nls2 (Grothendieck, 2013; Kates and Petzoldt, 2012) using the algorithm brute-force. The final coefficient estimates have been done with the function nlsLM (Elzhov et al, 2015) with the algorithm Levenberg-Marquardt.

Parameter estimation can be hard if some of them are correlated or have a huge range of possible values. To speed up or enable the algorithm to converge, the equation can be reformulated e.g. by multiplying one of the coefficients with the other and to use the coefficients in a transformed way like $C_0 = 1/c_0$ or $C_0 = exp(c_0)$.

Sometimes the data-point $t = 0$, $h = 0$ makes problems. In these cases this observation was not used. As most of the functions meet per definition this point, such a data reduction will not influence the coefficient estimates. The estimated standard-error will be influenced and needs to be calculated using all data. Alternatively this data point ($t = 0$; $h = 0$) could be removed for all functions, but some functions (e.g. Gompertz or Logit) do not hit this point. Functions which do not hit the data point ($t = 0$; $h = 0$) would benefit if this data point would be removed and the relation to other functions might be skewed.

All functions which are estimating the increment *ih* have been numerically integrated to a height over age curve. The starting point was $t = 0$ and $h = 0$. *ih* was not used as the slope of the tangent, it was used as the annual height increment. Coefficients have been estimated for least square fit of this height over age curve integral. With the present data (age and height) the annual height increment could be approximately estimated and used directly in these equations. Those coefficients are optimal to estimate, for a given age or height, the increment of the next year, but they need not automatically be optimal for the resulting growth curve over the whole observation time of the tree. As *ih* was used as the annual height increment, those functions need to estimate $ih > 0$ for the point where $h = 0$.

$$h = c_0 \cdot t \qquad Linear \tag{1.0}$$
$$h = c_0 \cdot t + c_1 \cdot t^2 \qquad Parabola \tag{1.1}$$
$$h = c_0 \cdot t + c_1 \cdot t^2 + c_2 \cdot t^3 \qquad Polynom3 \tag{1.2}$$
$$h = c_0 \cdot t + c_1 \cdot t^2 + c_2 \cdot t^3 + c_3 \cdot t^4 \qquad Polynom4 \tag{1.3}$$
$$h = c_0 \cdot t^{c_1} \qquad Allometric \tag{1.4}$$
$$h = c_0 \cdot t + c_1 \cdot t^2 + c_2 \cdot t^{c_3} \qquad PolyAllo \tag{1.5}$$

$$h = c_0 \cdot \mathrm{atan}^{c_1}(c_2 \cdot t^{c_3}) \qquad Atan \tag{2.0}$$
$$h = c_0 \cdot \tanh^{c_1}(c_2 \cdot t^{c_3}) \qquad Tanh \tag{2.1}$$
$$h = c_0 \cdot \ln^{c_1}(1 + c_2 \cdot t^{c_3}) \qquad Ln \tag{2.2}$$
$$h = c_0 \cdot \ln^{c_{1,fix}}(1 + e^{c_2} \cdot t^{c_3}) \qquad LnFixC1 \tag{2.3}$$
$$h = c_0 \cdot \ln^{c_1}(1 + e^{c_2} \cdot t^{c_{3,fix}}) \qquad LnFixC3 \tag{2.4}$$
$$h = c_0 \cdot \ln^{c_{1,fix}}(1 + e^{c_2} \cdot t^{c_{3,fix}}) \qquad LnFixC1C3 \tag{2.5}$$
$$h = c_0 \cdot \ln^{c_{1,fix}}(1 + e^{c_2} \cdot t^{c_{3,fix} + c_2 \cdot c_{4,fix}}) \qquad LnFixC1C3Fn \tag{2.6}$$

$$h = c_0 \cdot e^{c_1 \cdot t^{-1}} \qquad Terazaki \tag{3.0}$$
$$h = c_0 \cdot e^{c_1 \cdot t^{-1} + c_2 \cdot t^{-0.5}} \qquad TerazakiE1 \tag{3.1}$$
$$h = c_0 \cdot e^{c_1 \cdot t^{-0.5} + c_2 \cdot t^{c_3}} \qquad TerazakiE2 \tag{3.2}$$
$$h = c_0 \cdot e^{c_1 \cdot t^{c_2}} \qquad Korf \tag{3.3}$$
$$h = c_0 \cdot e^{c_1 \cdot e^{c_2 \cdot t}} \qquad Gompertz \tag{3.4}$$
$$h = c_0 \cdot e^{c_1 \cdot e^{c_2 \cdot t^{c_3}}} \qquad Sloboda \tag{3.5}$$
$$h = c_0 \cdot (e^{(c_1^2 - 1) \cdot e^{c_2 \cdot t^{c_3}}} - c_1 \cdot e^{c_2 \cdot t^{c_3}})^{c_1 * c_4 + 1} \qquad Grosenbaugh \tag{3.6}$$
$$h = c_0 \cdot e^{c_1 \cdot \ln(t) + c_2 \cdot \ln^2(t)} \qquad Korsun \tag{3.7}$$
$$h = c_0 \cdot e^{c_1 \cdot \ln(t) + c_2 \cdot \ln^2(t) + c_3 \cdot \ln^4(t)} \qquad KorsunE1 \tag{3.8}$$
$$h = c_0 \cdot e^{c_1 \cdot \ln(t) + c_2 \cdot \ln^{c_3}(t)} \qquad KorsunE2 \tag{3.9}$$

$$h = c_0(1 - e^{c_1 \cdot t}) \qquad Weber \tag{4.0}$$
$$h = c_0(1 - e^{c_1 \cdot t}) \cdot (1 - e^{c_2 \cdot t}) \qquad WeberE1 \tag{4.1}$$
$$h = c_0(1 - e^{c_1 \cdot t}) \cdot (1 - e^{c_2 \cdot t}) \cdot (1 - e^{c_3 \cdot t}) \qquad WeberE2 \tag{4.2}$$
$$h = c_0(1 - e^{c_1 \cdot t}) \cdot (1 - e^{c_2 \cdot t})^{c_3} \qquad WeberE3 \tag{4.3}$$
$$h = c_0 \cdot (1 - e^{c_1 \cdot t}) + c_2 \cdot (1 - e^{c_3 \cdot t}) \qquad \text{Kövessi1} \tag{4.4}$$
$$h = c_0\left(\frac{1 - e^{-c_1 \cdot t}}{c_1} - \frac{1 - e^{-c_2 \cdot t}}{c_2}\right) \qquad \text{Kövessi2} \tag{4.5}$$
$$h = c_0(1 - e^{c_1 \cdot t})^{c_2} \qquad Mitscherlich \tag{4.6}$$
$$h = c_0(1 - e^{c_1 \cdot t^{c_3}}) \qquad Fischer \tag{4.7}$$
$$h = c_0(1 - e^{c_1 \cdot t^{c_3}})^{c_2} \qquad Todorovic \tag{4.8}$$
$$h = c_0(1 - e^{c_1 \cdot t \cdot (1 - e^{c_2 \cdot t})}) \qquad Thomasius \tag{4.9}$$

$$h = \frac{c_0 \cdot t}{c_1 + c_2 \cdot \ln(t) + t} \qquad Hoßfeld3 \tag{5.0}$$
$$h = \frac{c_0}{1 + c_1 \cdot t^{-1} + c_2 \cdot t^{-2}} \qquad Hoßfeld1 \tag{5.1}$$
$$h = \frac{c_0}{1 + c_1 \cdot t^{-1} + c_2 \cdot t^{-2} + c_3 \cdot t^{-3}} \qquad Hoßfeld1E1 \tag{5.2}$$
$$h = \frac{c_0}{1 + c_1 \cdot t^{-1} + c_2 \cdot t^{c_3}} \qquad Yoschida \tag{5.3}$$
$$h = \frac{c_0}{1 + c_1 \cdot t^{c_2}} \qquad Hoßfeld4 \tag{5.4}$$
$$h = \frac{c_0}{(1 + c_1 \cdot t^{-1})^3} \qquad Strand \tag{5.5}$$
$$h = \frac{c_0}{(1 + c_1 \cdot t^{-1})^{c_3}} \qquad Levakovic88 \tag{5.6}$$
$$h = \frac{c_0}{(1 + c_1 \cdot t^{c_2})^{c_3}} \qquad Levakovic159 \tag{5.7}$$

$$h = \frac{c_0}{1 + c_1 \cdot e^{c_2 \cdot t}} \qquad Logit \tag{6.0}$$
$$h = c_4 + \frac{c_0}{1 + c_1 \cdot e^{c_2 \cdot t^{c_3}}} \qquad LogitE1 \tag{6.1}$$
$$h = c_4 + \frac{c_0}{1 + c_1 \cdot e^{c_2 \cdot t + c_3 \cdot t^2}} \qquad LogitE2 \tag{6.2}$$
$$h = \frac{c_0}{1 + c_1 \cdot e^{c_2 \cdot \ln(t) + c_3 \cdot \ln^2(t)}} \qquad LogitE3 \tag{6.3}$$
$$h = c_4 + \frac{c_0}{(1 + c_1 \cdot e^{c_2 \cdot t})^{c_3}} \qquad Nelder \tag{6.4}$$

$$h = c_0 \cdot (t/c_1)^{c_2 \cdot t^{c_3}} \qquad Siven \tag{7.0}$$
$$h = c_0 \cdot t^{c_1} \cdot e^{c_2 \cdot t} \qquad Gram \tag{7.1}$$
$$h = c_0 \cdot (1 - e^{-2\frac{t}{c_1}} \cdot (1 + 2 \cdot \frac{t}{c_1} + 2 \cdot (\frac{t}{c_1})^2)) \qquad Peschel \tag{7.2}$$

$$ih = c_0 \cdot e^{c_1 \cdot \ln(t) + c_2 \cdot \ln^2(t)} \qquad Backman \tag{8.0}$$
$$ih = c_0 \cdot e^{c_1 \cdot \ln(t) + c_2 \cdot \ln^2(t) \cdot c_3 \ln^4(t)} \qquad BackmanE1 \tag{8.1}$$
$$ih = c_0 \cdot t^{c_1} \cdot c_2^{-t} \qquad Koller \tag{8.2}$$
$$ih = c_0 \cdot t^{c_1} \cdot c_2^{-t^{c_3}} \qquad Levakovic147 \tag{8.3}$$

$$ih = c_0 \cdot (c_1 + h)^{c_2} \cdot (1 - (c_1 + h)/c_3)^{c_4} \qquad Hyperlog \tag{9.0}$$
$$ih = (c_1 + c_0 \cdot h^{c_2}) \cdot (1 - h/c_3)^{c_4} \qquad HyperlogE1 \tag{9.1}$$
$$ih = (c_0 + c_1 \cdot h^{c_2}) \cdot e^{c_3 \cdot h^{c_4}} \qquad HyperlogE2 \tag{9.2}$$
$$ih = (c_0 + c_1 \cdot h^{c_2}) \cdot (1/(1 + c_3 \cdot h^{c_4})) \qquad HyperlogE3 \tag{9.3}$$
$$ih = (c_0 + c_1 \cdot h^{c_2}) \cdot (1 - \tanh(c_3 \cdot h^{c_4})) \qquad HyperlogE4 \tag{9.4}$$
$$ih = (c_0 + c_1 \cdot h^{c_2}) \cdot (\frac{\pi}{2} - \mathrm{atan}(c_3 \cdot h^{c_4})) \qquad HyperlogE5 \tag{9.5}$$

2.4 Differences between observation and model

The RSME between the observed and the estimated height was calculated for each tree and each growth function with $RSME = \sqrt{\sum_1^{n_{obs}} (h_{obs} - h_{est})^2 / (n_{obs} - n_{coef})}$ where n_{obs} is the number of height observations, h_{obs} is the observed heights, h_{est} are the estimated heights and n_{coef} are the number of coefficients of the function which is estimating h_{est}.

The differences between the observed and the estimated height at the ages 10, 30, 70, 100 and 150 have been calculated for each tree and each growth function. In cases when height was not observed at a given age this height was interpolated between the enclosing two data points using a spline with the function splinefun with method monoH.FC.

The 0.05, 0.25, 0.5, 0.75 and 0.95 quantile for each function and age-step has been calculated. It has been tested if there is a trend of the difference of $h_{obs} - h_{est}$ over h_{est} by fitting a linear regression for each function and age-step by using the "High Breakdown and High Efficiency Robust Linear Regression" lmRob (Wang et al, 2014).

2.5 Height development

For each tree the height at age 10, 30, 70, 100, 150, 300 and 800 years was estimated with each function and the tree specific coefficients. The 0.05, 0.25, 0.5, 0.75 and 0.95 quantile of those height estimates has been calculated for each function and age-step.

3. Results

3.1 Root–mean–square error

For each tree its height at different ages is known. These heights are also estimated by the different functions with tree specific coefficients. For each tree the RSME between observed and estimated heights is calculated. Those RSME have been split into seven groups ranging from low to high deviation which contain approximately the same count of trees. The number of trees which have been classified into those groups are shown in figure 2 for data from Guttenberg and in figure 3 for the data from BFW for all growth functions. In addition the median of the RSME has been calculated and is given in the figures after the function name.

For the Guttenberg data-set many functions show a RSME below ±0.40 m and below ±0.50 m for the BFW trees. The group of the *HyperLog* function, which uses five coefficients, can describe the growth pattern of the trees very accurately. Also *Ln*, with only four coefficients can be found in the middle of this best performing group. This group follows the group of functions with four coefficients, where *Hoss1E1, Yoshida, Sloboda, Atan, Tanh, Levakovic159, BackmanE1, KorsunE1* and *Todorovic* are on the upper ranks. From the functions having only three parameters *Hoßfeld1, Hoßfeld4, Korsun, Levakovic88, Thomasius, Backman* and *Mitscherlich* show good results. The functions after *WeberE1* for the Guttenberg and those after *Korf* for the BFW data-set show not so good results.

By keeping the coefficient c_1 constant in the *Ln* function its RSME is increasing. For the BFW data-set it is better than the best three parameter function and for the Guttenberg data-set only Backman was slightly better. When keeping c_3 constant it is the best three parameter function for the Guttenberg data-set but for the BFW data it shows similar behaviour like the other three parameter functions. When keeping c_1 and c_3 constant its RSME again is increasing but it is better than Strand, the best two parameter function. Using a linear dependency of c_3 on c_2 shows only marginal improves of the result.

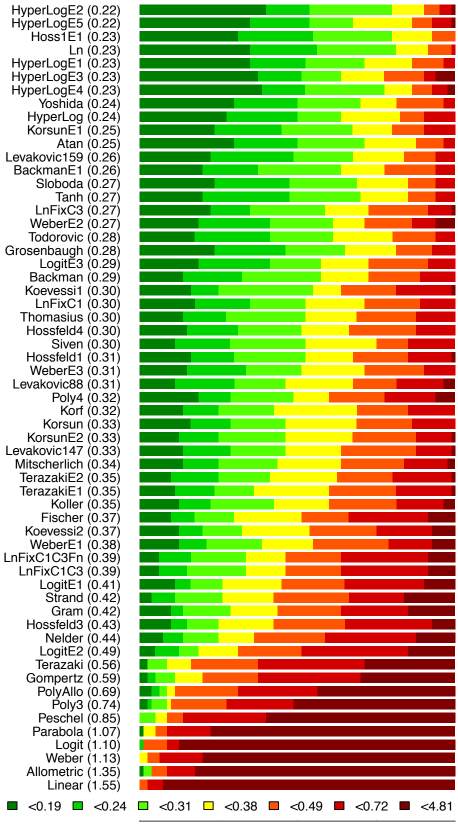

Figure 2: Distribution of the root–mean–square error for single trees in meter for the data from Guttenberg.

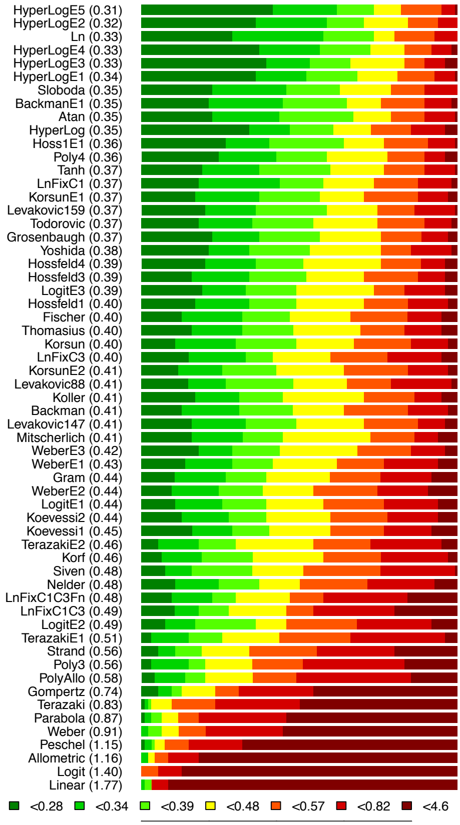

Figure 3: Distribution of the root–mean–square error single trees in meter for the data from BFW.

3.2 Height difference

For the ages 10, 30, 70, 100 and 150 the differences between observation and model are calculated. Their 0.05, 0.25, 0.5, 0.75 and 0.95 quantile of all trees for the specific growth function are calculated and shown in figure 4 for the Guttenberg and in figure 5 for the BFW trees. Many functions show deviations lower than 0.3m for 50% of the trees and in the range of 0.6m for 90% of the trees. The group of hyperlog functions show the smallest differences followed by Ln. This group is followed by the functions of *Hoss1E1, Sloboda, Grosenbaugh, Levakovic159, KorsunE1, Yoshida, Atan, Tanh* and *Todorovic*. From the three parameter functions *Hossfeld1, Backman, Thomasius* and *Levakovic88* showed the best results.

The height at age 150 shows a tendency to be underestimated from nearly all functions for the Guttenberg trees and to be overestimated for the BFW trees. For the BFW data-set a tendency of overestimation also for the age classes 100 years can be seen. The height at the age of 10 years is for both data-sets underestimated by many functions. An exception provides the group of the *hyperlog* functions which make bias-free estimates also for the height at age 10.

Keeping in *Ln* the coefficients c_1 or c_3 or both constant the differences between observed and estimated heights increases. The tendency to underestimate the height at age 100 and 150 for Guttenberg trees disappear. For the Guttenberg trees the form with constant c_1 and c_3 show only for the age 30 years a tendency of underestimation, for all other ages the estimates show a smaller bias but larger variation than the function without constant coefficients.

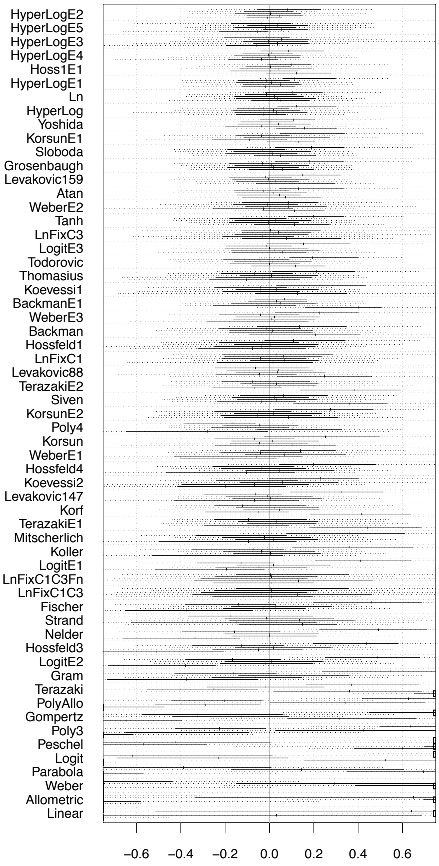

Figure 4: Height difference Guttenberg

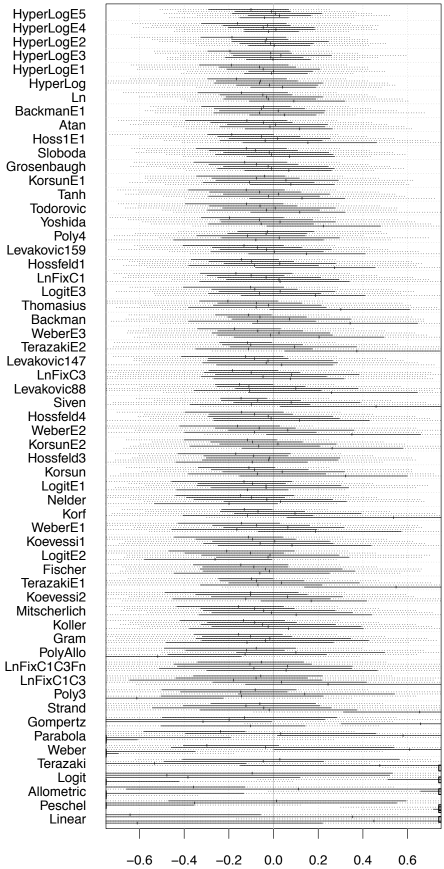

Figure 5: Height difference BFW

0.05 – 0.95 (dotted line), 0.25 – 0.75 (solid line) and 0.5 (vertical mark) quantile of $h_{observed} - h_{estimate}$ in meter for the ages 10 (lowest line), 30, 70, 100 and 150 (top line). Medians outside the shown range are indicated by dots at the border.

The slope of the difference between $h_{observed} - h_{estimate}$ plotted over $h_{estimate}$ is shown in figure 6 for Guttenberg trees and figure 7 for BFW trees. Nearly all functions show a considerable negative trend at age 10 which means that trees whose height is estimated low are in reality bigger and trees which are estimated to be big are in reality smaller. In the higher age classes a slightly positive trend can be seen in the Guttenberg data-set and a predominant negative trend in the BFW data-set. In the oldest age class (150 years) the estimated slope is close to zero but the RSME is bigger than in the other age classes. Here the ranking is not so clear as in the previous comparisons where the functions with many parameters are better than those with fewer. Nevertheless those with 4 to 5 parameters are on the better ranks. From those with three parameters *Hossfeld1*, *Backman*, *Thomasius* and *Levakovic88* showed good results. Keeping in *Ln* c_1 constant increases the slope deviation slightly.

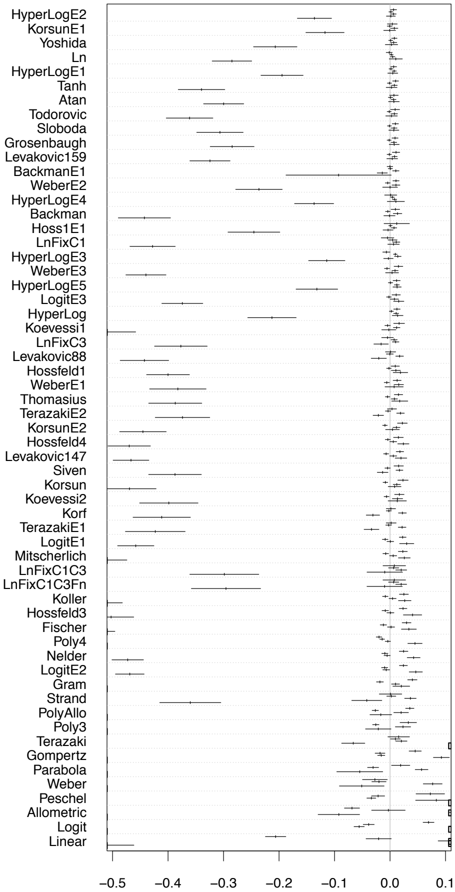

Figure 6: Trend on height difference with trees from Guttenberg

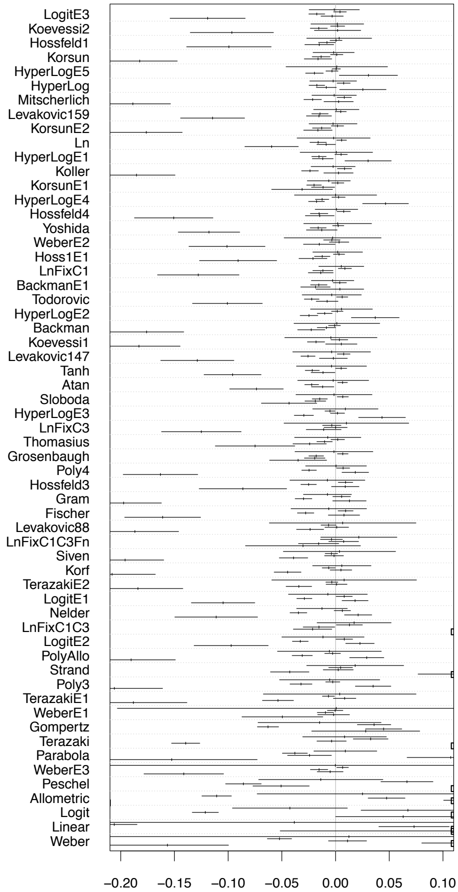

Figure 7: Trend on height difference with trees from BFW

Slope of $h_{observed} - h_{estimate}$ in meter plotted over $h_{estimate}$ for the ages 10 (lowest line), 30, 70, 100 and 150 (top line). The small horizontal line indicates the slope estimation and the solid horizontal line range of its root–mean–square error.

3.3 Height development

In figure 8 the 0.05, 0.25, 0.5, 0.75 and 0.95 quantile of the estimated height for Guttenberg trees for ages 10, 30, 70, 100, 150, 300 and 800 years and in figure 9 those for BFW trees are shown. The figures are sorted by the median height increment between 150 and 800 years. It could be seen that for the ages up to 150 years the heights are similar and differences between the functions compared to the observations could already be seen in figures 4 and 5. *HyperLog, Ln* and *Korf* still show some height increment at the higher ages. The estimated heights are for those functions still in a reasonable range of 40m to 80 m. On the lower end is the Logit function which shows never negative increments. Negative increments have been produced by the functions *Gram, Hoss1E1, Hossfeld3, Korsun, KorsunE1, KorsunE2, LogitE2, LogitE3, Parabola, Poly3, Poly4, PolyAllo, Siven, TerazakiE1* and *TerazakiE2*. The estimated height increments between 150 and 300 and also between 300 and 800 years are typical in the range of 0m to 10m which is approximately an annual height increment of 5 cm/year and 1 cm/year respectively. Keeping in *Ln* c_1 constant shows for the used data a slightly more bend, and keeping c_3 constant a slightly more straightened height development compared to the unrestricted function.

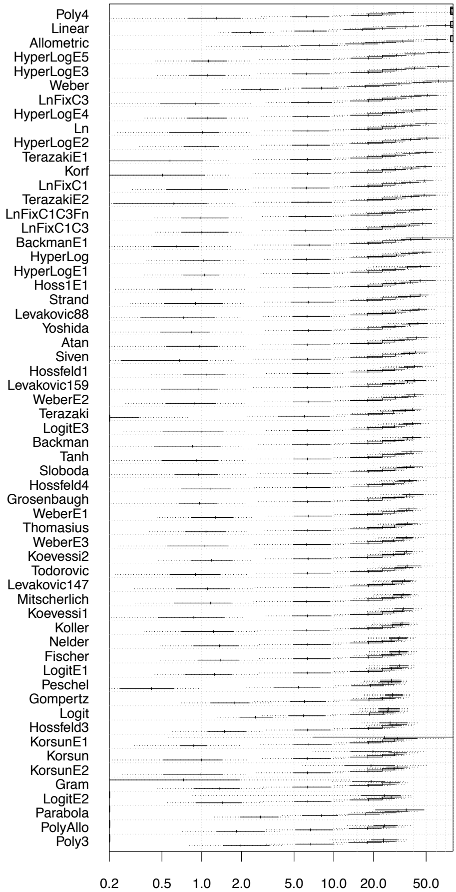

Figure 8: Height for ages 10, 30, 70, 100, 150, 300 and 800 with Guttenberg data.

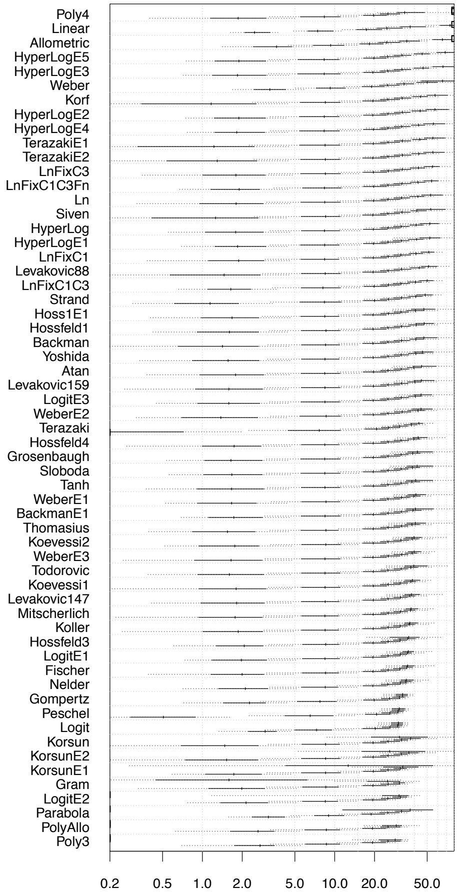

Figure 9: Height for ages 10, 30, 70, 100, 150, 300 and 800 with BFW data.

0.05 – 0.95 (dotted line), 0.25 – 0.75 (solid line) and 0.5 (vertical mark) quantile of $h_{estimate}$ in meter for the ages 10 (lowest line), 30, 70, 100, 150, 300 and 800 (top line). Medians outside the shown range are indicated by dots at the border.

4. Discussion

After comparing several functions it is not possible to select the best one. Especially those with more than three coefficients are flexible and show very similar shapes. For functions with less than three parameters their coefficients are easily estimated but they are not so flexible and show larger deviation from the observed growth pattern. Some functions with three parameters (*Hossfeld1, Backman, Thomasius, Levakovic88*) show acceptable fits. Functions with four parameters show good results. The tested functions with five parameters show only marginal improvements. Typically one of the functions with four parameters should be used. The final decision depends on whether the parameter estimation converges, the parameters can be interpreted directly (e.g. height at a specific age) or the function can be transformed to show either height depending on age or estimate age as a function of height.

Sometimes from growth functions, which describe height over age, the first derivation can be built or functions are directly formulated as differential equations which describe the slope of the increment at a given age or tree size. This slope is not identical with the annual increment. It seems admissible to use those differential functions to estimate differences instead of differentials. But then it is important to define for which time span this increment is estimated. If the time span of the observed increments is a multiple of the function time span, the function needs to be used recursively. Only for linear functions the increments can be adjusted with a multiplication of (time span function) / (time span observation). However, this simple increment adjustment can be used to find starting parameters either for the recursive function call or the integral function form.

Tests, whether differences between the functions are significant or not, have not been made, as the differences between the functions should have practical relevance and not statistical significance. If the difference between two height growth functions is statistically significant, but the difference is in a range of practical irrelevance, I would recommend to choose the simpler function. Also the criterion if the difference is significant or not depends beside the selected significance level on the attributes of subsample and its size. The influence of different samples on the function ranking by the applied methods is shown by using the two different height growth data sets. There is no need that a function showing good results on the used data sets is also good on other data sets. To find the functions which can follow the growth pattern in a good way on a different data set, all functions need to be tested. But it can be expected that functions with low deviations on the two data sets used here show also good behaviour on comparable data sets.

For the function $h = c_0 \cdot ln^{c_1}(1 + c_2 \cdot t^{c_3})$ it was tried to keep up to two coefficients constant for each data-set. This reduces the flexibility of the function. Keeping c_1 constant shows for both data-sets good results which are close to the best or better than the best three parameter function. Keeping c_3 constant shows good results for the Guttenberg trees. When holding c_1 and c_3 constant the deviation comes close to the range from acceptable to unacceptable but is still better than any of the tested two parameter functions. It could be expected, that keeping coefficients constant, but estimating it optimal for the used data-set, should result in better results than functions which do not have, or set some coefficients to a data independent fixed value.

If the linear intercorrelation between the coefficients makes it difficult, for the used algorithm, to find optimal parameters the formulations $h = c_0 \cdot \ln^{\frac{c_1}{c_2}}(1 + e^{c_2} \cdot t^{c_2 \cdot c_3})$, $h = c_0 \cdot \frac{\ln^{c_1}(1+c_2 \cdot t^{c_3})}{\ln^{c_1}(1+c_2 \cdot 100^{c_3})}$ or $h = c_0 \cdot \frac{\ln^{\frac{c_1}{c_2}}(1+e^{c_2} \cdot t^{c_2 \cdot c_3})}{\ln^{\frac{c_1}{c_2}}(1+e^{c_2} \cdot 100^{c_2 \cdot c_3})}$ could be tried, which have reduced the coefficient intercorrelation for the used data-sets. Those are only alternative formulations of the Ln function which might help the solver, but will not change the shape of the resulting curve. The last two forms make c_0 equal with the height at age 100.

5. Conclusions

From the examined 55 functions $h = c_0 \,.\, ln^{c_1}(1 + c_2 \,.\, t^{c_3})$ showed good behaviour and might be one which could be recommended. Only for the young ages, around 10 years their height is underestimated. The group of *Hyper-Logistic* functions dose not show this weakness. But they need an additional parameter and can directly show only the increments over height. The differences to functions like *Sloboda, Todorovic, Levakovic159, Grosenbaugh, Tanh* and *Atan* are marginal. From the three parameter functions *Hossfeld1, Backman, Thomasius and Levakovic88* showed good results.

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References

Backman G (1931) Das Wachstumsproblem. Ergebnisse der Physiologie 33(1):883–973, DOI 10.1007/BF02325889, URLhttp://dx.doi.org/10.1007/BF02325889

von Bertalanffy L (1934) Untersuchungen über die Gesetzlichkeit desWachstums. Wilhelm Roux' Archiv für Entwicklungsmechanik der Organismen 131(4):613–652, DOI 10.1007/BF00650112, URL http://dx.doi.org/10.1007/BF00650112

Blumberg AA (1968) Logistic growth rate functions. Journal of Theoretical Biology 21(1):42 – 44, DOI http://dx.doi.org/10.1016/0022-5193(68)90058-1, URL http://www.sciencedirect.com/science/article/pii/0022519368900581

Buis R (1991) On the generalization of the logistic law of growth. Acta Biotheoretica 39(3-4):185–195, DOI 10.1007/BF00114174, URL http://dx.doi.org/10.1007/BF00114174

Chapman S (1931) The absorption and dissociative or ionizing effect of monochromatic radiation in an atmosphere on a rotating earth part ii. grazing incidence. Proceedings of the Physical Society 43(5):483, URL http://stacks.iop.org/0959-5309/43/i=5/a=302

Chrobok V, Meloun M, imkov E (2004) Descriptive growth model of the height of stapes in the fetus: a histopathological study of the temporal bone. European Archives of

Oto-Rhino-Laryngology and Head & Neck 261(1):25–29, DOI 10.1007/s00405-003-0580-4, URL http://dx.doi.org/10.1007/s00405-003-0580-4

Elfving B, Kiviste A (1997) Construction of site index equations for pinus sylvestris l. using permanent plot data in Sweden. Forest Ecology and Management 98(2):125 – 134, DOI http://dx.doi.org/10.1016/S0378-1127(97)00077-7, URL http://www.sciencedirect.com/science/article/pii/S0378112797000777

Elzhov TV, Mullen KM, Spiess AN, Bolker B (2015) minpack.lm: R Interface to the Levenberg-Marquardt Nonlinear Least-Squares Algorithm Found in MINPACK, Plus Support for Bounds. URL http://CRAN.R-project.org/package=minpack.lm, r package version 1.1-9

Fisher RA, Tippett LHC (1928) Limiting forms of the frequency distribution of the largest or smallest member of a sample. Mathematical Proceedings of the Cambridge Philosophical Society 24:180–190, DOI 10.1017/S0305004100015681

Gompertz B (1825) On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. Philosophical Transactions of the Royal Society of London 115:pp. 513–583, URL http://www.jstor.org/stable/107756

Gottschalk PG, Dunn JR (2005) The five-parameter logistic: A characterization and comparison with the fourparameter logistic. Analytical Biochemistry 343(1):54 – 65, DOI http://dx.doi.org/10.1016/j.ab.2005.04.035, URL http://www.sciencedirect.com/science/article/pii/S0003269705003313

Gram JP (1879) Om konstruktion af normal-tilvæxotversigter. Tidsskrift for skovbrug (Gyldenfeld 1880 Ueber die Construktion von Normal-Zuwachsübersichten, Zeitschrift für Forst- und Jagdwesen 240 - 246) 3:207 – 270

Grosenbaugh LR (1965) Generalization and reparameterization of some sigmoid and other nonlinear functions. Biometrics 21(3):pp. 708–714, URL http://www.jstor.org/stable/2528551

Grothendieck G (2013) nls2: Non-linear regression with brute force. URL http://CRAN.R-project.org/package=nls2, r package version 0.2

von Guttenberg A (1915)Wachstum und Ertrag der Fichte im Hochgebirge. Deuticke

Hassell MP (1975) Density-dependence in single-species sopulations. Journal of Animal Ecology 44:283–295, DOI 10.2307/3863, URL http://dx.doi.org/10.2307/3863

Hill AV (1913) The combinations of haemoglobin with oxygen and with carbon monoxide. i. Biochemical Journal pp 471 – 480

Hoßfeld JW (1822) Mathematik für Forstmänner, Oekonomern und Cameralisten. Vierter Band, welcher die sphärisch Trigonometrie, die Stereometrie, Lehre von krummen Linien, Differezial- und Integralrechnung und die Momentenlehre enthält. Henningsche Buchhandlung

Johnson NO (1935) A trend line for growth series. Journal of the American Statistical Association 30(192):717–717, DOI 10.1080/01621459.1935.10503297, URL http://dx.doi.org/10.1080/01621459.1935.10503297, http://dx.doi.org/10.1080/01621459.1935.10503297

Kates L, Petzoldt T (2012) proto: Prototype object-based programming. URL http://CRAN.R-project.org/package=proto, r package version 0.3-10

Khamis A, Ismail Z, Haron K, Mohammed A (2005) Nonlinear growth models for modeling oil palm yield growth. Journal of Mathematics and Statistics pp 225–233

Koller FL (1886) Analytische Untersuchung über die Zuwachscurven. Österreichische Vierteljahresschrift für Forstwesen pp 32–51, 132–140

Korf V (1939) Prıspevek k matematické definici vzrustového zákona hmot lesních porostu. Lesnická práce pp 339–379

Korsun F (1935) Zivot norma´lnho porostu ve vzorcch. Lesnická práce pp 289–300

Kövessi F (1928) Az llnyek fejldse szablyossgnak a magyarzata, Erläuterungen der Gesetzmässigkeiten im Verlaufe der Lebenserscheinungen lebender Wesen. Matematikai 'es Term'eszettudom'anyi 'Ertest"o, Mathematischer und Naturwissenschaftlicher Anzeiger pp 652–664

Kövessi F (1929) Az llnyek fejldse szablyossgnak a magyarzata 4, Erläuterungen der Gesetzmässigkeiten im Verlaufe der Lebenserscheinungen lebender Wesen 4. Matematikai 'es Term'eszettudom'anyi 'Ertest"o, Mathematischer und Naturwissenschaftlicher Anzeiger pp 458–486

Koya PR, Goshu AT (2013) Generalized mathematical model for biological growths. Open Journal of Modelling and Simulation pp 42–53, DOI 10.4236/ojmsi.2013.14008

Kumaraswamy P (1980) A generalized probability density function for double-bounded random processes. Journal of Hydrology 46(12):79 – 88, DOI http://dx.doi.org/10.1016/0022-1694(80)90036-0, URL http://www.sciencedirect.com/science/article/pii/0022169480900360

Kuelka K, Maruk R (2015) Korfit: An efficient growth function fitting tool. Computers and Electronics in Agriculture 116:187 – 190, DOI http://dx.doi.org/10.1016/j.compag.2015.07.001, URL http://www.sciencedirect.com/science/article/pii/S0168169915001969

Levakovic A (1935) Analiticki izraz zu stastojinisku visinsku krivulju (Analytiscer Ausdruck für die Bestandeshöhenkurve). Glasnik za Sumske Pokuse Annales Pro Experimentis Foresticis 4:283 – 310

Mende W, Albrecht KF (2001) Beschreibung und Interpretation des Fichtenwachstums mit Hilfe des Evolon-Modells.

Forstwissenschaftliches Centralblatt vereinigt mit Tharandter forstliches Jahrbuch 120(1-6):53–67, DOI 10.1007/BF02796081, URL http://dx.doi.org/10.1007/BF02796081

Michaelis L, Menten ML (1913) Die Kinetik der Invertinwirkung. Biochemische Zeitschrift

Michailoff I (1943) Zahlenmäßiges Verfahren für die Ausführung der Bestandeshöhenkurven. Forstwissenschaftliches Centralblatt und Tharandter Forstliches Jahrbuch pp 273 – 279

Mitscherlich EA (1909) Das Gesetz des Minimums und das Gesetz des abnehmenden Bodenertrages. Landwirtschaftliche Jahrbücher

Mitscherlich EA (1919) Das Gesetz des Pflanzenwachstums. Landwirtschaftliche Jahrbücher

Morgan PH, Mercer LP, FlodinNW(1975) General model for nutritional responses of higher organisms. Proceedings of the National Academy of Sciences 72(11):4327–4331, DOI 10.1073/pnas.72.11.4327, URL http://www.pnas.org/content/72/11/4327.abstract, http://www.pnas.org/content/72/11/4327.full.pdf

Nelder JA (1961) The fitting of a generalization of the logistic curve. Biometrics 17(1):89–110, URL http://www.jstor.org/stable/2527498

Nelder JA (1962) 182. note: An alternative form of a generalized logistic equation. Biometrics 18(4):pp. 614–616, URL http://www.jstor.org/stable/2527907

Palah M, Tom M, Pukkala T, Trasobares A, Montero G (2004) Site index model for pinus sylvestris in north-east Spain. Forest Ecology and Management 187(1):35 – 47, DOI http://dx.doi.org/10.1016/S0378-1127(03)00312-8, URL http://www.sciencedirect.com/science/article/pii/S0378112703003128

Pearson K (1934) Tables of the Incomplete Beta-function. Great Britain. Dept. of Scientific and Industrial Research and University College, London. Dept. of Applied Statistics, Printed at the University Press and published by the Proprietors of Biometrika

Peschel W(1938) Die mathematischen Methoden zur Herleitung derWachstumsgesetze von Baum und Bestand und die Ergebnisse ihrer Anwendung. Tharandter Forstliches Jahrbuch pp 169–247

Pütter A (1920) Studien über physiologische Ähnlichkeit VI.Wachstumsähnlichkeiten. Pflüger's Archiv für die gesamte Physiologie des Menschen und der Tiere 180(1):298–340, DOI 10.1007/BF01755094, URL http://dx.doi.org/10.1007/BF01755094

R Core Team (2015) R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria, URL https://www.R-project.org/

Richards FJ (1959) A flexible growth function for empirical use. Journal of Experimental Botany 10(2):290–301, DOI 10.1093/jxb/10.2.290, URL http://jxb.oxfordjournals.org/content/10/2/290.abstract, http://jxb.oxfordjournals.org/content/10/2/290.full.pdf+html

Ricker W (1979) 11 growth rates and models. In: WS Hoar DR, Brett J (eds) Bioenergetics and Growth, Fish Physiology, vol 8, Academic Press, pp 677 – 743, DOI http://dx.doi.org/10.1016/S1546-5098(08)60034-5, URL http://www.sciencedirect.com/science/article/pii/S1546509808600345

Ricker WE (1954) Stock and recruitment. Journal of the Fisheries Research Board of Canada 11(5):559–623, DOI 10.1139/f54-039, URL http://dx.doi.org/10.1139/f54-039, http://dx.doi.org/10.1139/f54-039

Robertson TB (1908) On the normal rate of growth of an individual, and its biochemical significance. Archiv für Entwicklungsmechanik der Organismen 25(4):581–614, DOI 10.1007/BF02163864, URL http://dx.doi.org/10.1007/BF02163864

Schnute J (1981) A versatile growth model with statistically stable parameters. Canadian Journal of Fisheries and Aquatic Sciences 38(9):1128–1140, DOI 10.1139/f81-153, URL http://dx.doi.org/10.1139/f81-153, http://dx.doi.org/10.1139/f81-153

Schumacher FX (1939) A new growth curve and its application to timber-yield studies. Journal of Forestry pp 819 – 820

Sedmk R, Scheer L (2015) Properties and prediction accuracy of a sigmoid function of time-determinate growth. iForest - Biogeosciences and Forestry (5):631–637, DOI

10.3832/ifor1243-007, URL http://www.sisef.it/iforest/contents/?id=ifor1243-007, http://www.sisef.it/iforest/pdf/?id=ifor1243-007

Sivèn A (1896) Grundsätze zur Berechnung des Höhenwachstums der Nadelhölzer. Monatsschrift für das Forst- und Jagdwesen 18(2):91–94, DOI 10.1007/BF01841523, URL http://dx.doi.org/10.1007/BF01841523

Sloboda B (1971) Zur Darstellung vonWachstumsprozessen mit Hilfe von Differentialgleichungen erster Ordung, Mitteilungen der Baden-Baden-Württemberg Forstliche Versuchs- und Forschungsanstalt, Heft 32.

Smith JM, Slatkin M (1973) The stability of predator-prey systems. Ecology 54(2):384–391, URL http://www.jstor.org/stable/1934346

Stannard C, Williams A, Gibbs P (1985) Temperature/growth relationships for psychrotrophic food-spoilage bacteria. Food Microbiology 2(2):115 – 122, DOI http://dx.doi.org/10.1016/S0740-0020(85)80004-6, URL http://www.sciencedirect.com/science/article/pii/S0740002085800046

Strand L (1964) Numerical constructions of site-index curves. Forest Science 10(4):410–414, URL http://www.ingentaconnect.com/content/saf/fs/1964/00000010/00000004/art00005

Terazaki W (1915) Notes on the analytical interpretation of growth curves for single tree and stands and on application for the construction of yield table for sugi. Extracts from the Bulletin of the Forest Experiment Station Meguro, Tokyo pp 151 – 202

Thomasius H (1964) Düngung und Melioration in der Forstwirtschaft: Vorträge e. Tagung d. Arbeitsgemeinschaft Forstdüngung vom 29. - 30.10.1963 im Rahmen d. dies academicus d. Fak. f. Forstwirtschaft Tharandt d. Techn. Univ. Dresden. Deutsche Akademie der Landwirtschaftswissenschaften Berlin, Ost: Tagungsbericht, Fiedler, H.J.

Todorovic D (1961) Zakonitost organskog rastinja i njegova analiticka predstava. Glasnik Sumarskog fakulteta

Turner M, Blumenstein B, Sebaugh J (1969) 265 note: A generalization of the logistic law of growth. Biometrics 25(3):577–580, URL http://www.jstor.org/stable/2528910

Turner ME, Bradley EL, Kirk KA, Pruitt KM (1976) A theory of growth. Mathematical Biosciences 29(3):367 – 373, DOI http://dx.doi.org/10.1016/0025-5564(76)90112-7, URL http://www.sciencedirect.com/science/article/pii/0025556476901127

Upadhyay A, Eid T, Sankhayan PL (2005) Construction of site index equations for even aged stands of tectona grandis (teak) from permanent plot data in india. Forest Ecology and Management 212(13):14 – 22, DOI http://dx.doi.org/10.1016/j.foreco.2005.02.058, URL http://www.sciencedirect.com/science/article/pii/S0378112705001593

Wang J, Zamar R, Marazzi A, Yohai V, Salibian-Barrera M, Maronna R, Zivot E, Rocke D, Martin D, Maechler M, Konis K (2014) robust: Robust Library. URL http://CRAN.R-project.org/package=robust, r package version 0.4-16

Weber R (1891) Lehrbuch der Forsteinrichtung mit besonderer Berücksichtigung der Zuwachsgesetze der Waldbäume. Julius Springer Berlin

Yoshida M (1928) Do yowai tanjunrin ni okeru tan ki - ki no seicho kyokusen ni kansuru kenkyu, Untersuchungen über die Zuwachskurve eines Stammes und Bestandes im gleichaltrigen reinenWalde. Teikoku daigaku enshurin hokoku Tokyo, Mitteilungen der Universität für Wald der Kaiserlichen Universität Tokyo URL http://hdl.handle.net/2261/22467

Zeide B (1993) Analysis of growth equations. Forest Science 39(3):594–616, URL http://www.ingentaconnect.com/content/saf/fs/1993/Zhang L (1997) Cross-validation of non-linear growth functions for modelling tree heightdiameter relationships. Annals of Botany 79(3):251–257, DOI 10.1006/anbo.1996.0334, URL http://aob.oxfordjournals.org/content/79/3/251.abstract, http://aob.oxfordjournals.org/content/79/3/251.full.pdf+html